

# Witten's conjecture and Virasoro conjecture for genus up to two

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**ABSTRACT.** This is an expository paper based on the results in [16] and [12]. The main goal is to show how one may prove the following two conjectures *for genus up to two*:

- (1) Witten's conjecture on the relations between higher spin curves and Gelfand–Dickey hierarchy.
- (2) Virasoro conjecture for conformal semisimple Frobenius manifolds.

## 1. Introduction

### 1.1. The conjectures.

1.1.1. *Witten's conjecture.* E. Witten in 1990 made a striking conjecture between generating functions of intersection numbers on moduli spaces of stable curves and a  $\tau$ -function of KdV hierarchies [25]. This conjecture says that the following geometrically defined function

$$\tau^{pt}(t_0, t_1, \dots) = e^{\sum_{g=0}^{\infty} \hbar^{g-1} F_g^{pt}(t_0, t_1, \dots)}$$

is a  $\tau$ -function of the KdV hierarchy.<sup>1</sup> In the above formula,  $F_g^{pt}(t)$  is the generating function of (tautological) intersection numbers on the moduli space of stable curves of genus  $g$ . Moreover, from elementary geometry of moduli spaces, one easily deduces that  $\tau^{pt}$  satisfies an additional equation, called the *string equation*. It is known from the theory of KdV hierarchy that the string equation for the KdV (or in general KP) hierarchies uniquely determines a  $\tau$ -function parameterized by the Sato's grassmannian. This particular  $\tau$ -function will be called Witten–Kontsevich  $\tau$ -function and denoted  $\tau_{WK}$ . In other words,  $\tau_{WK} = \tau^{pt}$ . Often  $\tau^{pt}$  is used to emphasize its geometric nature and  $\tau_{WK}$  is used when the integrable system side is emphasized.

In 1991 Witten formulated a remarkable generalization of the above conjecture. He argued that an analogous generating function  $\tau^{r\text{-spin}}$  of the intersection numbers on moduli spaces of  $r$ -spin curves should be identified as a  $\tau$ -function of  $r$ -th Gelfand–Dickey ( $r$ -KdV) hierarchies [26]. When  $r = 2$ , this conjecture will

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Research partially supported by NSF..

<sup>1</sup> $\hbar$  is usually put to be 1.

*eventually* reduces to the previous one (after some work), as 2-KdV is the ordinary KdV.

The special case was soon proved by M. Kontsevich [14]. More recently a new proof was given by Okounkov–Pandharipande [22]. However, the generalized conjecture remains open up to this day.

It may be worth pointing out that the status of two conjectures was very different when they were first proposed. The 1990 conjecture was from the beginning formulated mathematically, using only well defined mathematical quantities. The 1991 conjecture, on the contrary, involves the concepts like moduli spaces of  $r$ -spin curves and their virtual fundamental classes (in modern terminologies) for which Witten offers only sketches of their construction. Perhaps the sharpest contrast lies in the fact that there were plenty of evidences supporting the 1990 conjecture, but virtually no evidences supporting 1991 conjecture beyond genus zero at the time they are formulated.

Throughout the years, T. Jarvis, and later joint by T. Kimura, A. Vaintrob, and A. Polishchuk, T. Mochizuki, have clarified the foundational issues. In particular, Jarvis–Kimura–Vaintrob [13] established the genus zero case of the conjecture; Mochizuki and Polishchuk independently established the following property for  $\tau^{r\text{-spin}}$ :

**THEOREM 1.** [21] [24] *All tautological equations hold for  $F_g^{r\text{-spin}}$ .*

In fact,  $F_g^{r\text{-spin}}$  satisfies all “expected functorial properties”, similar to the axioms formulated by Kontsevich–Manin in the Gromov–Witten theory. Note that the term *tautological equations* usually mean the relations of tautological (Chow) classes on moduli spaces of curves. Here they are used for the relations of the classes on the moduli spaces of spin curves, obtained by the forgetful map  $\overline{M}_{g,n}^{1/r} \rightarrow \overline{M}_{g,n}$ . Later, they are also used for the induced relations in the Gromov–Witten theory.

However, Riemann’s trichotomy of Riemann surfaces has taught us that things are very different in genus one and at higher genus. Our Main Theorem therefore provides a solid confirmation for Witten’s 1991 conjecture, covering one example ( $g = 1$  and  $g = 2$ ) for the other two cases in the trichotomy. In fact, this work starts as a project trying to understand this conjecture at higher genus.

For more background information about Witten’s conjecture, the readers are referred to Witten’s original article [26] and the paper [13] by Jarvis–Kimura–Vaintrob, both well-written. In the remaining of this article, “Witten’s conjecture” means the 1991 conjecture if not otherwise specified.

**1.1.2. Virasoro conjecture.** In 1997 another generalization of Witten’s 1990 conjecture was proposed by T. Eguchi, K. Hori and C. Xiong [5]. Witten’s 1990 conjecture has an equivalent formulation:  $\tau^{pt}$  is annihilated by infinitely many differential operators  $\{L_n^{pt}\}$ ,  $n \geq -1$ , satisfying the Virasoro relations

$$[L_m, L_n] = (m - n)L_{m+n}.$$

Eguchi–Hori–Xiong, and S. Katz, managed to formulate a conjecture for any projective smooth variety  $X$ , generalizing the above assertion. Namely, they found the formulas of  $\{L_n^X\}$  for  $n \geq -1$ , satisfying Virasoro relations and conjectured that

$$L_n^X \tau^X(t) = 0, \quad \text{for } n \geq -1.$$

In the above formula,

$$\tau^X(t) := e^{\sum_{g=0}^{\infty} \hbar^{g-1} F_g^X(t)},$$

and  $F_g^X(t)$  is the generating function of genus  $g$  Gromov–Witten invariants with descendants for the projective manifold  $X$ . This conjecture is commonly referred to as the Virasoro conjecture.

Eguchi–Hori–Xiong was able to give strong evidences for their conjecture at genus zero. Later X. Liu and G. Tian [20] established the genus zero case. Using a very different method, B. Dubrovin and Y. Zhang established the genus one case of Virasoro conjecture for *conformal* semisimple Frobenius manifolds.<sup>2</sup>

The recent major developments are Givental's proof of Virasoro conjecture for toric Fano manifolds [9] and Okounkov–Pandharipande's proof of Virasoro conjecture for algebraic curves [23].

**1.2. Givental's theory.** A. Givental introduces a completely new approach to Gromov–Witten theory in a series of papers [8, 9, 11], dating back to August 2000. It is beyond our ability to summarize Givental's theory in a few paragraphs. In the following we will restrict ourselves to some highlights of his theory, mainly for the purpose of fixing the notations and putting the theorems in a framework. The details can be found in [17].

The essence of his theory is a construction of a “combinatorial model” of higher genus invariants via graphic enumeration, with the information of edges coming from the underlying semisimple Frobenius manifold (i.e. genus zero theory) and information of vertices from  $\tau^{pt}$ . Formulaically, given a semisimple Frobenius manifold  $H$  of dimension  $N$ , he defines an operator  $\hat{O}_H = \exp(\partial_H)$ . Givental's  $\tau$ -function is defined to be

$$(1) \quad \tau_G^H := e^{\sum_{g=0}^{\infty} \hbar^{g-1} G_g^H(t^1, \dots, t^N)} := \hat{O}_H \prod_{i=1}^N \tau_{\text{WK}}(t^i, \hbar).$$

In fact, the operator  $\hat{O}_H$  is a special kind of operator and belongs to quantized twisted loop group, which will be discussed in Section 3.<sup>3</sup> The Feynman rules then dictate a formula for  $G_g$ . When the Frobenius manifold comes from geometry, i.e.  $H = QH^*(X)$ , Givental conjectures that his combinatorial model is the same as the geometric model. That is  $G_g^H = F_g^X$  when  $H = QH^*(X)$ .

What makes Givental's model especially attractive are the facts that

- (1) it works for any semisimple Frobenius manifolds
- (2) it enjoys properties often complementary to the geometric theory.

Thanks to (1), one also has a Givental's model for the Frobenius manifolds  $H_{A_{r-1}}$  of the miniversal deformation space of  $A_{r-1}$  singularity. It turns out that this Frobenius manifold is isomorphic to the Frobenius manifold defined by the genus zero potential of  $r$ -spin curves. Furthermore, Givental has recently proved

THEOREM 2. [10]  $\tau_G^{H_{A_{r-1}}}$  is a  $\tau$ -function of  $r$ -KdV hierarchy.

As in the case of the ordinary KdV, it is easy to show that both  $\tau_G^{H_{A_{r-1}}}$  and  $\tau^{r\text{-spin}}$  satisfy the additional string equation. Therefore, in order to prove Witten's conjecture, one only has to answer the following question positively,

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<sup>2</sup>The definition of Frobenius manifolds in this article does not require existence of an Euler field, which is assumed in Dubrovin's definition. Dubrovin's definition will be referred to as *conformal Frobenius manifold* instead.

<sup>3</sup> $\hat{O}_H$  is actually not really in the quantized twisted loop group, but in its completion. We will ignore the difference in this article.

QUESTION 1. *Is  $G_g^{H_{A_{r-1}}} = F_g^{r\text{-spin}}$ ?*

As examples for (2),  $\tau_G^H$  satisfies Virasoro conjecture and  $\tau^X$  satisfies the tautological relations almost by definitions. In the second case, one notes that if the theory is defined geometrically, one can pull-back the relations on moduli spaces of curves to moduli spaces of maps. In the first case, one defines the Virasoro operators for semisimple Frobenius manifold  $H$  by

$$L_n^H(t) := \hat{O} \prod_i L_n^{pt}(t_i) \hat{O}^{-1},$$

it is obvious that  $L_n^H$  also satisfy Virasoro relations. One immediately gets Virasoro conjecture for  $H$  by Kontsevich's theorem. It is also true that  $L_n^H = L_n^X$  when the semisimple Frobenius manifold  $H$  comes from quantum cohomology of  $X$ , i.e.  $H = QH^*(X)$ .

However, the converse statements pose nontrivial challenges.

QUESTION 2. *Does the tautological relations hold for  $G_g$ ?*

QUESTION 3. *Does Virasoro conjecture hold for  $\tau^X$ ?*

An obvious, and indeed very good, strategy to answer all the above three questions is to answer a generalized version of Question 1:

QUESTION 4. *Is  $G_g = F_g$ ? That is, does the combinatorial construction coincide with the geometric one when both are available?*

A positive answer to Question 4 obviously answer all three questions at once. Interesting enough, the solution to Question 4 turns out to be closely related to, and in many cases equivalent to, the solutions of Questions 2 and 3 by some uniqueness theorems.

For example, the answer to Question 3 is equivalent to that to Question 4 in the geometric Gromov–Witten theory by a result of Dubrovin and Zhang [4]. They proved that Virasoro conjecture plus  $(3g-2)$ -jet conjecture (actually a theorem of E. Getzler [7] in geometric Gromov–Witten theory and of Givental in the context of semisimple Frobenius manifolds) uniquely determines  $\tau$ -function for any semisimple Frobenius manifold. It is expected that  $\tau^{r\text{-spin}}$  will also satisfy the  $(3g-2)$ -jet property.

As for Question 2, the equivalence is established by some uniqueness theorems in genus one and two. Similar phenomena are “expected” to hold in higher genus as well, although there is no hard evidences at this moment. In genus one, Dubrovin and Zhang made the following important observation about the uniqueness theorem.

LEMMA 1. [2] *The genus one descendent potentials for any semisimple Frobenius manifolds  $H$  are uniquely determined, up to linear combination of canonical coordinates, by genus zero potentials, genus one topological recursion relations, and genus one Getzler's equation.*

*Furthermore, if  $H$  is conformal, then the genus one potential is uniquely determined up to constant terms.*

The proof of this fact goes as follows. First, genus one TRR guarantees that the descendent invariants are uniquely determined by primary invariants. Second, genus one Getzler's equation, when written in canonical coordinates  $u^i$ , is equal to

$\frac{\partial^2 F_1}{\partial u^i \partial u^j} = B_{ij}$  where  $B_{ij}$  involves only genus zero invariants. Moreover, the conformal structure determined by a linear vector field (Euler field), uniquely determines the linear term.

The uniqueness theorem in genus two, proved by X. Liu, is much more involved:

**THEOREM 3.** [18] *The genus two descendent potentials for any conformal semisimple Frobenius manifolds are uniquely determined up to constants by genus two equations by Mumford, Getzler [6] and Belorousski–Pandharipande (BP) [1].*

It is worth noting that whether this uniqueness theorem, or any weaker version, holds for non-conformal semisimple Frobenius manifolds remains unknown.

**1.3. Statements of the main results.** By Lemma 1, the differentials of the genus one potentials are equal,  $dG_1 = dF_1$ , if  $G_1$  satisfies genus one TRR and genus one Getzler's equation, plus some initial condition to fix the constant terms. The positive answer to Question 2 in genus one and therefore all other questions are proved in [12]

**THEOREM 4.** [12]  *$dG_1 = dF_1$  for all semisimple Frobenius manifolds.*

This theorem generalizes the earlier results by Dubrovin–Zhang for *conformal* semisimple Frobenius manifolds in [2].

**THEOREM 5.** [16]  *$G_2$  satisfies genus two tautological equations by Mumford, Getzler and BP.*

These two theorems, combined with the above results, immediately implies

**MAIN THEOREM.** [16] *Witten's conjecture and Virasoro conjecture for conformal semisimple Frobenius manifolds hold up to genus two.*

**1.3.1. Main ideas involved in the proofs.** To prove Theorem 4 and 5, note that

- $\tau_G = \hat{O} \prod_i \tau^{pt}(t^i)$ .
- $\tau^{pt}(t)$  satisfies all tautological equations.

Therefore, in order to prove  $G_g$  satisfies tautological equations, one only has to prove that these equations are invariant under the action of  $\hat{O}$ . This is the approach taken in [12] and [16].

**REMARK 1.** There are other possible approaches to this problem. Our earlier approach in [15] reduces the checking of Theorem 5 to a complicated, but finite-time checkable, identities. Nevertheless, it lacks the underlying simplicity of this approach.

After this result was announced, X. Liu informed us (and later posted in arxiv [19]) that he was also able to prove the genus two Virasoro conjecture for the conformal semisimple Gromov–Witten theory by reducing it to some complicated identities which he was able to check by hand *and* by a Mathematica programs.

**ACKNOWLEDGEMENT.** I would like to thank B. Dubrovin, T. Jarvis, T. Kimura, X. Liu, Y. Ruan and A. Vaintrob for useful discussions and communications. I am also thankful to E. Getzler for pointing out an inaccurate statement in an earlier version. Many ideas of this work comes from my collaborations with A. Givental [12] and R. Pandharipande [17]. It is a great pleasure to thank both of them.

## 2. Frobenius manifolds

**2.1. Givental's theory of formal Frobenius manifolds.** Let  $H$  be a complex vector space of dimension  $N$  with a distinguished element  $\mathbf{1}$ . Let  $(\cdot, \cdot)$  be a  $\mathbb{C}$ -bilinear metric on  $H$ , i.e. a nondegenerate symmetric  $\mathbb{C}$ -bilinear form. Let  $\mathcal{H}$  denote the (infinite dimensional) complex vector space  $H((z^{-1}))$  consisting of Laurent formal series in  $1/z$  with vector coefficients.<sup>4</sup> Introduce the symplectic form  $\Omega$  on  $\mathcal{H}$ :

$$\Omega(f, g) = \frac{1}{2\pi i} \oint (f(-z), g(z)) dz.$$

The polarization  $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$  by the Lagrangian subspaces  $\mathcal{H}_+ = H[z]$  and  $\mathcal{H}_- = z^{-1}H[[z^{-1}]]$  provides a symplectic identification of  $(\mathcal{H}, \Omega)$  with the cotangent bundle  $T^*\mathcal{H}_+$ .

Let  $\{\phi_\mu\}$  be an *orthonormal* basis of  $H$ . Introduce Darboux coordinates  $\{p_k^\mu, q_k^\mu\}$ ,  $k = 0, 1, 2, \dots$  and  $\mu = 1, \dots, N$ , compatible with this polarization, so that

$$\Omega = \sum dp_k^\mu \wedge dq_k^\mu.$$

An  $H$ -valued Laurent formal series can be written in this basis as

$$\dots + (p_1^1, \dots, p_1^N) \frac{1}{(-z)^2} + (p_0^1, \dots, p_0^N) \frac{1}{(-z)} + (q_0^1, \dots, q_0^N) + (q_1^1, \dots, q_1^N)z + \dots$$

To simplify the notations,  $p_k$  will stand for the vector  $(p_k^1, \dots, p_k^N)$  and  $p^\mu$  for  $(p_0^\mu, p_1^\mu, \dots)$ .

Let  $A(z)$  be an  $\text{End}(H)$ -valued Laurent formal series in  $z$  satisfying

$$(A(-z)f(-z), g(z)) + (f(-z), A(z)g(z)) = 0,$$

then  $A(z)$  defines an infinitesimal symplectic transformation

$$\Omega(Af, g) + \Omega(f, Ag) = 0.$$

An infinitesimal symplectic transformation  $A$  of  $\mathcal{H}$  corresponds to a quadratic polynomial  $P(A)$  in  $p, q$

$$P(A)(f) := \frac{1}{2}\Omega(Af, f).$$

( $A$  is a symplectic vector fields on the symplectic vector space  $(\mathcal{H}, \Omega)$ , and the relation between the function  $P(A)$  and vector field  $A$  is  $dP(A) = i_A\Omega$ .) For example, if  $\dim H = 1$  and  $A(z) = 1/z$ , then

$$P(z^{-1}) = -\frac{q_0^2}{2} - \sum_{m=0}^{\infty} q_{m+1}p_m.$$

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<sup>4</sup>Different completions of this spaces are used in different places, but this subtlety will be ignored in the present article.

**2.2. Lagrangian cones.** Let  $F_0(t)$  be a formal series in  $t$ , where  $t = (t_0, t_1, t_2, \dots)$  is related to  $q = (q_0, q_1, q_2, \dots)$  through the following change of variables:

$$\sum_{k=0}^{\infty} q_k z^k =: q(z) = t(z) - z\mathbf{1} := -z\mathbf{1} + \sum_{k=0}^{\infty} t_k z^k.$$

Thus the formal function  $F_0(t(z))$  near  $t = 0$  becomes a formal function  $F_0(q)$  on the space  $\mathcal{H}_+$  near the point  $q(z) = -z$ . This convention is called the *dilaton shift*.

In the Gromov–Witten theory,  $F_0(t)$  is the genus zero descendent potential. It satisfies many properties due to the geometry of the moduli spaces. Three classes of partial differential equations are most relevant. They are called the *Topological Recursion Relations* (TRR), the *String Equation* (SE) and the *Dilaton Equation* (DE):

$$\begin{aligned} (\text{DE}) \quad & \frac{\partial F_0(t)}{\partial t_1^1}(t) = \sum_{n=0}^{\infty} \sum_{\nu} t_n^{\nu} \frac{\partial F_0(t)}{\partial t_n^{\nu}} - 2F_0(t), \\ (\text{SE}) \quad & \frac{\partial F_0(t)}{\partial t_0^1} = \frac{1}{2}(t_0, t_0) + \sum_{n=0}^{\infty} \sum_{\nu} t_{n+1}^{\nu} \frac{\partial F_0(t)}{\partial t_n^{\nu}}, \\ (\text{TRR}) \quad & \frac{\partial^3 F_0(t)}{\partial t_{k+1}^{\alpha} \partial t_l^{\beta} \partial t_m^{\gamma}} = \sum_{\mu} \frac{\partial^2 F_0(t)}{\partial t_k^{\alpha} \partial t_0^{\mu}} \frac{\partial^3 F_0(t)}{\partial t_0^{\mu} \partial t_l^{\beta} \partial t_m^{\gamma}} \end{aligned}$$

for all  $\alpha, \beta, \gamma$  and all  $k, l, m \geq 0$ . Note that  $t_k^l$  denote the dual coordinates of the vectors  $\mathbf{1}z^k$ .

Denote by  $\mathcal{L}$  the graph of the differential  $dF_0$ :

$$\mathcal{L} = \{(p, q) \in T^*\mathcal{H}_+ : p = d_q F_0\}.$$

It is considered as a formal germ at  $q = -z$  (i.e.  $t = 0$ ) of a Lagrangian section of the cotangent bundle  $T^*\mathcal{H}_+$  and can therefore be considered as a formal germ of a Lagrangian submanifold in the symplectic loop space  $(\mathcal{H}, \Omega)$ .

**THEOREM 6.** [11] *The function  $F_0$  satisfies TRR, SE and DE if and only if the corresponding Lagrangian submanifold  $\mathcal{L} \subset \mathcal{H}$  has the following properties:*

- (1)  $\mathcal{L}$  is a Lagrangian cone with the vertex at the origin.
- (2) The tangent spaces  $L_f = T_f \mathcal{L}$  satisfy  $zL_f \subset L_f$  (and therefore  $\dim L_f/zL_f = \dim \mathcal{H}_+/z\mathcal{H}_+ = \dim H$ ).
- (3)  $zL_f \subset \mathcal{L}$ .
- (4) The same  $L_f$  is the tangent space to  $\mathcal{L}$  not only along the line of  $f$  but also at all smooth points in  $zL_f \subset \mathcal{L}$ .

One may rephrase the above properties by saying that  $\mathcal{L}$  is a cone ruled by the isotropic subspaces  $zL$  varying in a  $\dim H$ -parametric family with the tangent spaces along  $zL$  equal to the same Lagrangian space  $L$ . This in particular implies that the family of  $L$  generates a *variation of semi-infinite Hodge structures* in the sense of S. Barannikov, i.e. a family of semi-infinite flags

$$\cdots zL \subset L \subset z^{-1}L \cdots$$

satisfying the Griffiths integrability condition.

We note the following theorem

**THEOREM 7.** [11] *A Lagrangian cone satisfying the above conditions defines a formal germ of a Frobenius manifold plus a given calibration, in the sense of [9], and vice versa.*

Although two formulations are equivalent, the Lagrangian cone formulation is much more transparent and geometric. One can say that the Lagrangian cone formulation is the geometrization of the equations SE, DE, and TRR. Moreover, these properties are formulated in terms of the symplectic structure  $\Omega$  and the operator of multiplication by  $z$ . Hence it does not depend on the choice of the polarization. This shows that the system DE+SE+TRR has a huge symmetry group.

**DEFINITION.** Let  $\mathcal{L}^{(2)}GL(H)$  denote the *twisted loop group* which consists of  $\text{End}(H)$ -valued formal Laurent formal series  $M(z)$  in the indeterminate  $z^{-1}$  satisfying  $M^*(-z)M(z) = \mathbf{1}$ . Here  $^*$  denotes the adjoint with respect to  $(\cdot, \cdot)$ .

The condition  $M^*(-z)M(z) = \mathbf{1}$  means that  $M(z)$  is a symplectic transformation on  $\mathcal{H}$ .

**COROLLARY 1.** *The action of the twisted loop group preserves the class of the Lagrangian cones  $\mathcal{L}$  satisfying (1-4) and, generally speaking, yields new generating functions  $F_0$  which satisfy the system DE+SE+TRR whenever the original one does.*

### 3. Higher genus and quantization

To quantize an infinitesimal symplectic transformation, or its corresponding quadratic hamiltonians, we recall the standard Weyl quantization. A polarization  $\mathcal{H} = T^*\mathcal{H}_+$  on the symplectic vector space  $\mathcal{H}$  (the phase space) defines a configuration space  $\mathcal{H}_+$ . The quantum “Fock space” will be a certain class of functions  $f(\hbar, q)$  on  $\mathcal{H}_+$  (containing at least polynomial functions), with additional formal variable  $\hbar$  (“Planck’s constant”). The classical observables are certain functions of  $p, q$ . The quantization process is to find for the classical mechanical system on  $\mathcal{H}$  a “quantum mechanical” system on the Fock space such that the classical observables become operators on the Fock space. In particular, the classical hamiltonians  $h(q, p)$  on  $\mathcal{H}$  are quantized to be differential operators  $\hat{h}(q, \frac{\partial}{\partial q})$  on the Fock space.

In the above Darboux coordinates, the quantization  $P \mapsto \hat{P}$  assigns

$$(2) \quad \begin{aligned} \hat{1} &= 1, \quad \hat{p}_k^i = \sqrt{\hbar} \frac{\partial}{\partial q_k^i}, \quad \hat{q}_k^i = q_k^i / \sqrt{\hbar}, \\ (p_k^i p_l^j)^\wedge &= \hat{p}_k^i \hat{p}_l^j = \hbar \frac{\partial}{\partial q_k^i} \frac{\partial}{\partial q_l^j}, \\ (p_k^i q_l^j)^\wedge &= q_l^j \frac{\partial}{\partial q_k^i}, \\ (q_k^i q_l^j)^\wedge &= \hat{q}_k^i \hat{q}_l^j / \hbar, \end{aligned}$$

Note that one often has to quantize symplectic instead of infinitesimal symplectic transformations. Following the common practice in physics, we define

$$(3) \quad (e^A)^\wedge := e^{(A)^\wedge},$$

for the symplectic transformation  $e^{A(z)}$  in the twisted loop group.

When one restricts the attention to semisimple Frobenius (formal) manifolds, the situation is even simpler. Let  $H_0 = \mathbb{C}^N$  be the Frobenius manifold such that the orthonormal basis  $\{\phi_\mu\}$  form idempotents of the product

$$\phi_\mu * \phi_\nu = \delta_{\mu\nu} \phi_\mu.$$

Or in terms of  $F_0$

$$F_0(t) = \frac{1}{6} \sum_\mu (t_0^\mu)^3.$$

Givental's  $\tau$ -function (1) for  $H_0$  becomes  $\tau_G^{H_0}(q) = \prod_\mu \tau^{pt}(q^\mu)$ . It follows from Givental's theory that  $\tau_G^H$  for any semisimple Frobenius manifold  $H$  is

$$\tau_G^H = \hat{O}_H \prod \tau_G^{H_0}.$$

Furthermore,  $\hat{O}_H$  is actually an element in the quantized twisted loop group.<sup>5</sup> Therefore, we conclude that

**MAIN LEMMA.** *In order to show that a set of tautological equations holds for  $G_g$ , it suffices to show that this set of tautological equations is invariant under quantized loop group action.*

This lemma is our main technical tool to prove Theorem 4 and 5 and therefore Main theorem. In fact, in order to prove the invariance of the tautological equations, it is enough to prove the *infinitesimal* invariance of the tautological equations by the definition (3).

**REMARK 2.** Morally, one can consider the space of all semisimple Frobenius manifold as a homogeneous space of quantized twisted loop groups. However, there are many issues, including the issue of completion alluded before, which make this assertion invalid.

#### 4. Invariance of tautological equations under the action of the twisted loop groups

**4.1. Quantization of twisted loop groups.** The twisted loop group is generated by “lower triangular subgroup” and the “upper triangular subgroup”. The lower triangular subgroup consists of  $\text{End}(H)$ -valued formal formal series in  $z^{-1}$   $S(z^{-1}) = e^{s(z^{-1})}$  satisfying  $S^*(-z)S(z) = \mathbf{1}$  or equivalently

$$s^*(-z^{-1}) + s(z^{-1}) = 0.$$

The upper triangular subgroup consists of the regular part of the twisted loop groups  $R(z) = e^{r(z)}$  satisfying  $R^*(-z)R(z) = \mathbf{1}$  or equivalently

$$(4) \quad r^*(-z) + r(z) = 0.$$

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<sup>5</sup>This statement is not completely correct as the quantized twisted loop group is not a group. Nevertheless, it is a good heuristic picture.

For illustration, let us work out the quantization of the upper triangular subgroups. The quantization of  $r(z)$  is

$$\begin{aligned}\hat{r}(z) &= \sum_{l=1}^{\infty} \sum_{n=0}^{\infty} \sum_{i,j} (r_l)_{ij} q_n^j \partial_{q_{n+l}^i} \\ &+ \frac{\hbar}{2} \sum_{l=1}^{\infty} \sum_{m=0}^{l-1} (-1)^{m+1} \sum_{ij} (r_l)_{ij} \partial_{q_{l-1-m}^i} \partial_{q_m^j}.\end{aligned}$$

Let  $\frac{d\tau_G}{d\epsilon_r} = \hat{r}(z)\tau_G$ . Then

$$\begin{aligned}(5) \quad \frac{d}{d\epsilon_r} \langle \partial_{k_1}^{i_1} \partial_{k_2}^{i_2} \dots \rangle &= \sum_{l=1}^{\infty} \sum_{n=0}^{\infty} \sum_{i,j} (r_l)_{ij} q_n^j \langle \partial_{n+l}^i \partial_{k_1}^{i_1} \dots \rangle \\ &+ \sum_{l=1}^{\infty} \sum_{i,a} (r_l)_{ii_a} \langle \partial_{k_a+l}^i \partial_{k_1}^{i_1} \dots \partial_{k_a}^{\hat{i}_a} \dots \rangle \\ &+ \frac{1}{2} \sum_{l=1}^{\infty} \sum_{m=0}^{l-1} (-1)^{m+1} \sum_{ij} (r_l)_{ij} \partial_{k_1}^{i_1} \partial_{k_2}^{i_2} \dots (\langle \partial_{l-1-m}^i \rangle \langle \partial_m^j \rangle).\end{aligned}$$

For  $g \geq 1$

$$\begin{aligned}(6) \quad &\frac{d\langle \partial_{k_1}^{i_1} \partial_{k_2}^{i_2} \dots \rangle_g}{d\epsilon_r} \\ &= \sum_{l=1}^{\infty} \sum_{n=0}^{\infty} \sum_{i,j} (r_l)_{ij} q_n^j \langle \partial_{n+l}^i \partial_{k_1}^{i_1} \dots \rangle_g \\ &+ \sum_{l=1}^{\infty} \sum_{i,a} (r_l)_{ii_a} \langle \partial_{k_a+l}^i \partial_{k_1}^{i_1} \dots \partial_{k_a}^{\hat{i}_a} \dots \rangle_g \\ &+ \frac{1}{2} \sum_{l=1}^{\infty} \sum_{m=0}^{l-1} (-1)^{m+1} \sum_{ij} (r_l)_{ij} \langle \partial_{l-1-m}^i \partial_m^j \partial_{k_1}^{i_1} \partial_{k_2}^{i_2} \dots \rangle_{g-1} \\ &+ \frac{1}{2} \sum_{l=1}^{\infty} \sum_{m=0}^{l-1} (-1)^{m+1} \sum_{ij} \sum_{g'=g}^g (r_l)_{ij} \partial_{k_1}^{i_1} \partial_{k_2}^{i_2} \dots (\langle \partial_{l-1-m}^i \rangle_{g'} \langle \partial_m^j \rangle_{g-g'}).\end{aligned}$$

#### 4.2. Invariance theorems.

**THEOREM 8.** ( $S$ -invariance theorem) *All tautological relations are invariant under action of lower triangular subgroups of the twisted loop groups.*

This theorem combined with observations in [12] and the result from [9], which shows that there exists a lower triangular element  $S$  such that  $\hat{S}\tau^X$  is the generating function of “ancestors”, implies that one may assume that  $\tau^X$  is the ancestor potential and that  $q_0 = 0$  in the proof of  $R$ -invariance.

Let us use the term “genus zero relations” for genus zero dilaton equation, string equation, and TRR, the term “genus one relations” for genus one Getzler’s equation and genus one TRR, the term “genus two relations” for genus two equations by Mumford, Getzler, and BP.

**THEOREM 9. ( $R$ -invariance theorem)** *The union of the sets of genus  $g'$  relations for  $g' \leq g$  is invariant under the action of upper triangular subgroup, for  $g \leq 2$ .*

In fact, a stronger “filtered” statement holds. We will state the genus two part:

- (1) The combination of genus zero relations, genus one relations and Mumford’s equation is  $R$ -invariant.
- (2) The combination of genus zero relations, genus one relations and genus two Mumford’s and Getzler’s equations is  $R$ -invariant.

**REMARK 3.**  $R$ -invariance theorem is expected to hold for all  $g$ . This will be discussed in a separate paper.

### References

- [1] P. Belorousski, R. Pandharipande, *A descendent relation in genus 2*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 29 (2000), no. 1, 171–191.
- [2] B. Dubrovin, Y. Zhang, *Bi-Hamiltonian hierarchies in 2D topological field theory at one-loop approximation*, Comm. Math. Phys. 198 (1998), no. 2, 311–361.
- [3] B. Dubrovin, Y. Zhang, *Frobenius manifolds and Virasoro constraints*, Selecta Math. (N.S.) 5 (1999), no. 4, 423–466.
- [4] B. Dubrovin, Y. Zhang, *Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov - Witten invariants*, math.DG/0108160.
- [5] T. Eguchi, K. Hori, C. Xiong, *Quantum cohomology and Virasoro algebra*, Phys. Lett. B 402 (1997), no. 1-2, 71–80.
- [6] E. Getzler, *Topological recursion relations in genus 2*, Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), 73–106, World Sci. Publishing, River Edge, NJ, 1998.
- [7] E. Getzler, *The jet-space of a Frobenius manifold and higher-genus Gromov-Witten invariants*, math.AG/0211338.
- [8] A. Givental, *Semisimple Frobenius structures at higher genus*, Internat. Math. Res. Notices 2001, no. 23, 1265–1286.
- [9] A. Givental, *Gromov-Witten invariants and quantization of quadratic Hamiltonians*, Dedicated to the memory of I. G. Petrovskii on the occasion of his 100th anniversary. Mosc. Math. J. 1 (2001), no. 4, 551–568, 645.
- [10] A. Givental,  *$A_{n-1}$  singularities and  $nKdV$  hierarchies*, math.AG/0209205.
- [11] A. Givental, *Symplectic geometry of Frobenius structures*, math.AG/0305409
- [12] A. Givental, Y.-P. Lee, In preparation.
- [13] T. Jarvis, T. Kimura, A. Vaintrob, *Moduli spaces of higher spin curves and integrable hierarchies*, Compositio Math. 126 (2001), no. 2, 157–212.
- [14] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys. 147 (1992), no. 1, 1–23.
- [15] Y.-P. Lee, *Witten’s conjecture, Virasoro conjecture, and semisimple Frobenius manifolds*, draft, Aug 2002.
- [16] Y.-P. Lee, *Witten’s conjecture, Virasoro conjecture, and invariance of tautological relations*, math.AG/0311100. Updated version available at <http://www.math.utah.edu/~yplee/research/>.
- [17] Y.-P. Lee, R. Pandharipande, *Frobenius manifolds, Gromov-Witten theory, and Virasoro constraints*, in preparation.
- [18] X. Liu, *Quantum product on the big phase space and the Virasoro conjecture*, Adv. Math. 169 (2002), no. 2, 313–375.
- [19] X. Liu, *Genus-2 Gromov-Witten invariants for manifolds with semisimple quantum cohomology*, math.DG/0310410.
- [20] X. Liu, G. Tian, *Virasoro constraints for quantum cohomology*, J. Differential Geom. 50 (1998), no. 3, 537–590.
- [21] T. Mochizuki, *The virtual class of the moduli stack of  $r$ -spin curves*, preprint available from <http://www.math.ias.edu/~takuro/list.html>.
- [22] A. Okounkov, R. Pandharipande, *Gromov-Witten theory, Hurwitz numbers, and Matrix models, I*, math.AG/0101147

- [23] A. Okounkov, R. Pandharipande, *Virasoro constraints for target curves*, math.AG/0308097.
- [24] A. Polishchuk, *Witten's top Chern class on the moduli space of higher spin curves*, math.AG/0208112.
- [25] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, Surveys in differential geometry (Cambridge, MA, 1990), 243–310, Lehigh Univ., Bethlehem, PA, 1991.
- [26] E. Witten, *Algebraic geometry associated with matrix models of two-dimensional gravity*, Topological methods in modern mathematics (Stony Brook, NY, 1991), 235–269, Publish or Perish, Houston, TX, 1993.

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